Efficient Orbit Integration by Scaling for Kepler Energy Consistency

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We propose a new approach to integrate the quasi-Keplerian orbits numerically. First, the method integrates the usual equation of motion and the time evolution of the Kepler energy simultaneously;

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{dt}} = \frac{-\mu}{r^3}\mathbf{x} + \mathbf{a}, \quad \frac{\mathrm{d}K}{\mathrm{dt}} = \mathbf{v} \cdot \mathbf{a}.$$

Here $r \equiv |\mathbf{x}|$, $\mathbf{a}$ is the perturbing acceleration, $K \equiv T+U$ is the Kepler energy, $T \equiv \mathbf{v}^2/2$, and $U \equiv -\mu/r$. Next, we adjust $\mathbf{x}$ and $\mathbf{v}$ by a scale transformation

$$(\mathbf{x}, \mathbf{v}) \to (s\mathbf{x}, s\mathbf{v})$$

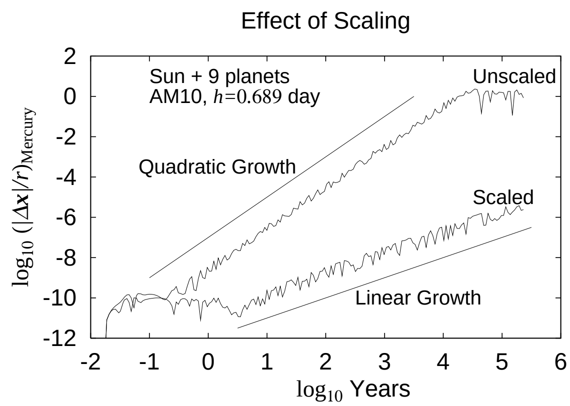
in order to satisfy the Kepler energy relation at every integration step. The scale factor, s, is determined by solving an associated cubic equation

$$Ts^3 - Ks + U = 0$$

with the help of the Newton method starting from $s = (2T - U)/(3T - K)$.

In treating multiple bodies, the Kepler enegies are integrated for each body and the scale factors are adjusted separately. The implementation of the new method is simple, the additional cost of computation is little, and its applicability is so wide that allowed are the integrations with variable step size.

Numerical experiments showed that the scaling reduces the integration error drastically. In case of pure Keplerian orbits, the truncation error grows only linearly with respect to time. When the perturbations exist, a component growing in a quadratic or higher power of time appears but its magnitude is reduced significantly when compared with the case without scaling.



The figure shows the relative position errors of Mercury obtained by the new and existing methods in the simultaneous numerical integration of the Sun and nine major planets for 2.4×10^5 years. The error of the unscaled integration increases in proportion to the square of time while that of the scaled one seems to grow linearly for this period.

The manner of decrease of the main error component roughly follows the power law of the strength of the perturbing acceleration where the power index depends on the type of perturbation from $5/4$ of the air drag, 2 of third body and J_2, and $5/2$ of general relativity. Also the method seems to suppress the accumulation of round-off errors although the details remain to be investigated.

In conclusion, the new approach provides a fast and high precision device to simulate the orbital motions of major and minor planets, natural and artificial satellites, comets, and space vehicles at negligible increase of computational cost.